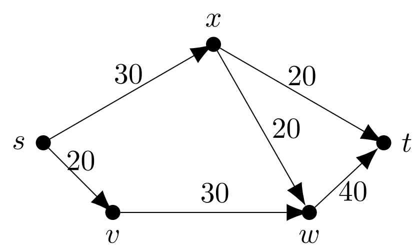
Chapter 6.2 Flows in Networks

Original application (product of cold war)

Suppose CCCP and USA start the war. How quickly can CCCP move tank from storage s in Siberia to the target t (battle ground in Europe)? The tanks are moved on a railroad. Every link gives the capacity how many tank a day can be transported.



1: How many tanks per day can be delivered to the battleground? Is the solution unique?

Solution: 50 tanks. The solution is not unique.

Problem: A (directed) graph G, source s, sink t, capacities $u : E(G) \to \mathbb{R}^+$.
Network is (G, u, s, t).

Input: Network (G, u, s, t)
Output: s-t-flow of maximum value

s-t-**flow** f is a function $f : E(G) \to \mathbb{R}^+$ such that $f(e) \leq u(e)$ for every $e \in E$.
Value of f is $\sum_{\vec{sv}\in E} f(sv) - \sum_{\vec{vs}\in E} f(vs)$ i.e. leaving − entering to s.

2: How does f look around one vertex of the network? (what axioms must f satisfy?)

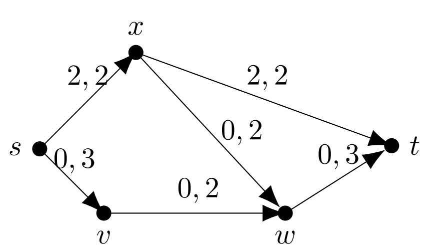
Solution: Flow conservation rule: For all $v \in V \setminus \{s, t\}$:

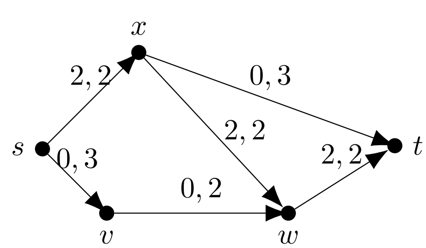
$$\sum_{xv\in E} f(xv) = \sum_{vx\in E} f(xv).$$

and for s, t it satisfies:

$$\sum_{sx\in E} f(sx) - \sum_{xs\in E} f(xs) = \sum_{xt\in E} f(xt) - \sum_{tx\in E} f(tx).$$

3: How do we improve these flows to be maximum? The description on each edge is the value of f, u.





Solution: Path s, v, w, t can be increased by 2. Path s, v, w, x, t can be increased by two.

4: After the improvement, how do you argue that nobody can further improve the flow?

Solution: Consider set $A = \{s, v\}$. The capacity of edges going from A is 4. Hence no flow can route more than 4 through the network.

Let $A \subset V(G)$ be such that $s \in A$ and $t \notin A$. Use $\delta^+(A)$ to denote set of edges $\overrightarrow{xy}$, where $x \in A$ and $y \notin A$ (edges leaving A). Use $\delta^-(A)$ to denote set of edges $\overrightarrow{xy}$, where $x \notin A$ and $y \in A$ (edges entering A).
Capacity of s-t-cut A is $\sum_{e\in\delta^+(A)} u(e)$.

5: Prove that for A and any flow f holds
(a) $\text{value}(f) = \sum_{e\in\delta^+(A)} f(e) - \sum_{e\in\delta^-(A)} f(e)$
(b) $\text{value}(f) \le \sum_{e\in\delta^+(A)} u(e)$

Solution: (a) Use conservation of flow at vertices in A.

$$\begin{aligned}\text{value}(f) &= \sum_{e\in\delta^+(s)} f(e) - \sum_{e\in\delta^-(s)} f(e)\\ &= \sum_{v\in A}\left(\sum_{e\in\delta^+(v)} f(e) - \sum_{e\in\delta^-(v)} f(e)\right)\\ &= \sum_{e\in\delta^+(A)} f(e) - \sum_{e\in\delta^-(A)} f(e)\end{aligned}$$

(b) clearly $\sum_{e\in\delta^+(A)} f(e) - \sum_{e\in\delta^-(A)} f(e) \le \sum_{e\in\delta^+(A)} u(e)$ since $f(e) \le u(e)$.

This proves the *obvious* observation that maximum flow cannot exceed capacity of minimum cut.

Notice in 3. we were improving flow be reducing the flow on $\overrightarrow{xw}$. We "sent flow in the opposite direction".

For a digraph G, define $\overleftrightarrow{G}$ by adding for every edge e also its **reverse** $\overleftarrow{e}$.

For f and u define **residual capacities** $u_f : E(\overleftrightarrow{G}) \to \mathbb{R}^+$

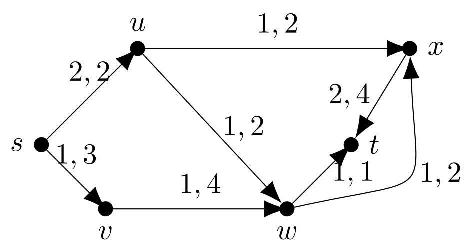
$$u_f(e) = u(e) - f(e) \qquad\qquad u_f(\overleftarrow{e}) = f(e)$$

Residual capacities ... how much extra we can send in each direction.

Residual graph G_f is obtained from $\overleftrightarrow{G}$ by removing edges $e \in E(\overleftrightarrow{G})$ with $u_f(e) = 0$.

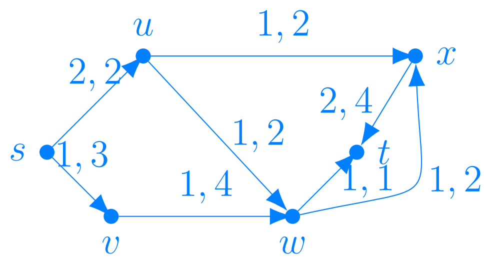
Augmenting path is an s-t path in G_f.

6: Construct the residual graph for



and find an augmenting path and increase the flow using the augmenting path.

Solution:



TODO: Too much drawing :-(

7: How do we update (i.e., **augment** the flow f) using an augmenting path in $\overleftrightarrow{G}$?

Solution: If flow is increasing on edge e by γ, then decrease $u_f(e)$ by γ and increase $u_f(\overleftarrow{e})$ by γ.

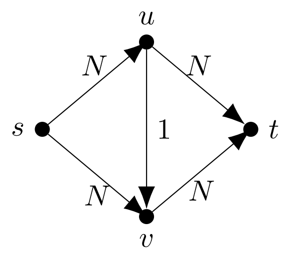
8: How to create an algorithm?

Solution: Keep finding augmenting paths and using them. The value of the value will be increasing.

Ford-Fulkerson Algorithm
Input: Network (G, u, s, t).
Output: an s-t-flow f of maximum value

1. $f(e) = 0$ for all $e \in E(G)$
2. while f-augmenting path P in G_f exists:
3. compute $\gamma := \min_{e \in E(P)} u_f(e)$
4. augment f along P by γ (as much as possible)

9: The algorithm might pick the augmenting path P poorly. At worst, how many iterations might the following network take?



(N is a big integer. Try to trick the algorithm to do many steps by picking unlucky P.)

Solution: Always use the edge uv with capacity 1. The augmentation will be always by one. So $2N$ iterations. The running time depends on u.

10: Show that s-t-flow f is maximum if and only if there is no f-augmenting path. (That is, Ford-Fulkerson algorithm is correct.)

Solution: If there is an augmenting path, then the flow can increase. If there is no augmenting path, then the set of vertices s reachable from s in the reduced graph G_f form a cut. Recall

$$\text{value}(f) = \sum_{e \in \delta^+(A)} f(e) - \sum_{e \in \delta^-(A)} f(e).$$

Since there is no augmenting path,

$$\sum_{e \in \delta^-(A)} f(e) = 0 \qquad \sum_{e \in \delta^+(A)} f(e) = \sum_{e \in \delta^+(A)} f(e).$$

This proves the maximality of the flow.

The question gives proof to
Theorem (Ford Fulkerson) Maximum value of an s-t-flow equals minimum capacity of an s-t-cut.

11: If $c : E \to \mathbb{Z}$, is it true that the flow produced from Ford-Fulekrson is integral and that the algorithm finishes in a finite time?

Solution: Yes, the augmenting is always by an integer. Every augmentation raises the value of the flow by at least one. This gives finiteness.

This proves
Theorem Dantzig Fulkerson: If the capacities are integral, then there exists an integral maximum flow.

12: If capacities are integral, is it true that every maximum flow is integral?

Solution: No - build network that has small cut and 2 paths to/or from it. See the following example with f, u on edges in this order.